Tutorial Notes 9

1. Suppose that C is a smooth, simple, closed curve in the plane that encloses Ω. Let $\bar{x}$ be the x-coordinate of the centroid of Ω. Show that
$$\frac{1}{2}\int_C x^2\,\mathrm{d}y = -\int_C xy\,\mathrm{d}x = \frac{1}{3}\int_C x^2\,\mathrm{d}y - xy\,\mathrm{d}x = |\Omega|\bar{x}.$$
Solutions:
By Green's theorem,
$$\begin{aligned}\frac{1}{2}\int_C x^2\,\mathrm{d}y &= \int_\Omega x\,\mathrm{d}x\,\mathrm{d}y\\ -\int_C xy\,\mathrm{d}x &= \int_\Omega x\,\mathrm{d}x\,\mathrm{d}y\\ \frac{1}{3}\int_C x^2\,\mathrm{d}y - xy\,\mathrm{d}x &= \int_\Omega x\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$
Moreover,
$$\int_\Omega x\,\mathrm{d}x\,\mathrm{d}y = |\Omega|\bar{x}.$$

2. (a) Suppose that f satisfies
$$\Delta f = f_{xx} + f_{yy} = 0.$$
Prove that
$$\int_C f_y\,\mathrm{d}x - f_x\,\mathrm{d}y = 0$$
for C to which Green's theorem applies.

(b) Conversely, assume that
$$\int_C f_y\,\mathrm{d}x - f_x\,\mathrm{d}y = 0$$
for every C to which Green's theorem applies. Prove that
$$\Delta f = 0.$$

Solutions:

(a) By Green's theorem,
$$\int_C f_y\,\mathrm{d}x - f_x\,\mathrm{d}y = \int_\Omega (-f_{yy} - f_{xx})\,\mathrm{d}x\,\mathrm{d}y = 0,$$
supposing that C encloses Ω.

(b) Take C to be circles $\partial B(x, r)$. By Green's theorem, we have

$$\int_{\partial B(x,r)} f_y \, \mathrm{d}x - f_x \, \mathrm{d}y = \int_{B(x,r)} (-f_{yy} - f_{xx}) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

Since

$$\lim_{r \to 0} \frac{\int_{B(x_0,r)} \Delta f \, \mathrm{d}x}{|B(x_0, r)|} = \Delta f(x_0),$$

it holds that

$$\Delta f = 0.$$

3. Let $f(x, y) = \log(x^2 + y^2)$.

(a) Let C be the circle $x^2 + y^2 = a^2$. Evaluate

$$\int_C \nabla f \cdot n \, \mathrm{d}s.$$

(b) Let Γ be a smooth, simple, closed in the plane that does not pass through $(0, 0)$. Evaluate

$$\int_\Gamma \nabla f \cdot n \, \mathrm{d}s.$$

Solutions:

$$\nabla f = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right)$$

and

$$\Delta f = 0.$$

(a) Note that $n = (x/a, y/a)$. The integral is

$$\int_C \frac{2}{a} \, \mathrm{d}s = 4\pi.$$

(b) Suppose that Γ enclose Ω. If the origin is outside Ω, then by Green's theorem and $\Delta f = 0$, the integral vanishes. If the origin is inside Ω, then Green's theorem can not be applied since there is a singularity at the origin. Our approach is to remove the origin by a small ball $B(0, \varepsilon)$. Applying Green's theorem to $\Omega \setminus B(0, \varepsilon)$, since $\Delta f = 0$,

$$\int_\Gamma \nabla f \cdot n \, \mathrm{d}s = \int_{\partial B(0,\varepsilon)} \nabla f \cdot n \, \mathrm{d}s.$$

By (a), the integral is 4π.

Remark 1

The idea used by the exercise is useful and interesting. When we calculate an integral of some vector field and we find that, for example, the curl or divergence vanish, we may use, for example, Green's theorem to convert the curve to another "good" curve.

For example, when we calculate the integral

$$\int_\Gamma \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

where Γ is the ellipse $x^2/a^2 + y^2/b^2 = 1$, if we directly use the parametrization of the ellipse, the calculation is not easy, but if we note that the curl of the vector field vanishes, then we can convert the ellipse to a circle around the origin which is easy to be calculated.

4. (a) Suppose that the curve $(f(u), g(u))$ where $g(u) > 0$ and $a \le u \le b$ is revolved about x-axis. Find the parametrization of the resulting surface of revolution.

   (b) Find the parametrization of the surface obtained by revolving the curve $x = y^2$, $y \ge 0$, about the x-axis.

   Solutions:

   (a) The parametrization is $(f(u), g(u)\cos v, g(u)\sin v)$.

   (b) The parametrization is $(y^2, y\cos v, y\sin v)$.